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ABSTRACT

This booklet is a collection of 45 problems that have appeared in the Problem Department of the journal "School Science and Mathematics." The booklet is divided into four parts. The first part presents 20 cryptarithms, puzzles in which letters or symbols are substituted for the digits in an arithmetical calculation, that require knowledge of arithmetic and some degree of ingenuity to solve. The second part is called arithmetical pastimes and presents 25 problems requiring some facility with algebra. The two sections include hints and/or answers to the problems. Complete solutions as they were appeared in the journal are reprinted in the final two parts of the booklet. The number of the problem and month and year of publication are given for each problem. (MDH)

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SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION

Topics for Teachers Series—Number 1

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Cryptarithms and Other Arithmetical Pastimes

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SE 053 133

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SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION

Topics for Teachers Series

Number 1

CRYPTARITHMS

AND

OTHER ARITHMETICAL PASTIMES

N. J. Kuenzi

and

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University of Wisconsin, Oshkosh

May 1979

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FOREWORD

This first volume in the Topics for Teachers Series was prepared by two persons who have edited the "Mathematics Problems Department" of School Science and Mathematics for a number of years. We would like to express our appreciation to N. J. Kuenzi and Bob Prielipp for selecting the items for this volume: they gave careful attention to meeting the needs of teachers at various levels for problems which are interesting to students and which promote mathematical aims.

Suggestions for other volumes in the Series are welcomed by the Publications Committee of SSMA.

Arthur F. Coxford
Marilyn N. Suydam
SSMA Publications Committee

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PREFACE

For over a half a century the Problem Department has been a continuing feature of School Science and Mathematics. Many hundreds of people have submitted solutions to problems which have been posed on its pages. Cryptarithms and Other Arithmetical Pastimes is a tribute to their efforts. In addition, it is an attempt to gather together a collection of problems which can be employed for enrichment purposes in the teaching of mathematics. We hope that you will use it not only for your own enjoyment, but also as a way to introduce mathematical problem solving to others.

As its name implies this booklet is essentially divided into two parts. The first part deals with cryptarithms (puzzles in which letters or symbols are substituted for the digits in an arithmetical calculation). In most instances, a knowledge of arithmetic (along with some degree of ingenuity) will be sufficient to solve these problems. The second portion is called arithmetical pastimes. Here some facility with algebra is generally required.

Included with each of the two sections are hints and/or answers. If one encounters difficulty with a problem, he or she may wish to consult this part of the booklet. Finally, solutions are reprinted substantially as they appeared in the issues of School Science and Mathematics. The notation (2202, November 1950), for example, following the number of a problem indicates that this problem is number 2202 in the numbering system used by the Problem Department and that a solution to it may be found in the November 1950 issue of School Science and Mathematics.

We hope that you will find these problems both interesting and challenging.

Good luck!

N. J. KUENZI BOB PRIELIPP

CRYPTARITHMS

Cryptarithms are puzzles in which letters or symbols are substituted for the digits in an arithmetical calculation. Although these kinds of problems have existed for several centuries, no one seems to be certain when they were first devised. Generally only a knowledge of arithmetic and some degree of ingenuity are required to solve a cryptarithm. Perhaps the best way to exemplify this is to consider a typical problem of this kind.

EXAMPLE. Consider the following addition cryptarithm, where each letter represents a decimal digit and different letters represent different digits:

$$\begin{array}{r} S\ E\ N\ D \\ M\ \emptyset\ R\ E \\ \hline M\ \emptyset\ N\ E\ Y \end{array}$$

Determine the digits.

(To avoid confusion between the digit "0" and the letter "O", we shall denote the letter by "$\emptyset$" and the digit by "0".)

SOLUTION. Generally the left-most digit of an expression is not allowed to be zero. Hence $S \neq 0$ and $M \neq 0$. Because $M \neq 0$ it is easily seen that M in the total must be 1, since the sum of the column containing S and M cannot reach 20. If the M in this column is replaced by a 1, how can we make this column total at least 10 to provide the required carry of 1? Only by making S 9 or 8. In either case $\emptyset$ must be zero. (The sum of S and M, plus a possible carry, can only be 10 or 11, but we cannot use 1 for $\emptyset$ because M is 1.) If $\emptyset$ is zero, then in the column containing E and $\emptyset$ we cannot have a total of 10 or more so there will be no 1 to carry over to the next column. Thus S must be 9. Since $E + \emptyset$ plus a possible carry of 1 equals N and $\emptyset$ is zero, N must be 1 greater than E. (Recall that E and N must be different.) Hence $N = E + 1$ and $N + R$ plus a possible carry of 1 equals $E + 10$. Subtracting, we obtain R plus a possible carry of 1 is 9. We cannot let $R = 9$ because we already have $S = 9$. Therefore $R = 8$ and there is a carry of 1 from the right-hand column. The column containing D and E must total at least 12 since Y cannot be 1 or zero. What values can we give D and E to reach this total? We have already used 9 and 8. The

only digits left that are high enough are 7, 6 and 7, 5. But one of these has to be E and N = E + 1. Thus E must be 5, N must be 6, and D must be 7. Then Y turns out to be 2. The complete restoration is:

$$\begin{array}{r} 9\,5\,6\,7 \\ +\;1\,0\,8\,5 \\ \hline 1\,0\,6\,5\,2 \end{array}$$

Note that solving a cryptarithm includes finding all of its solutions and describing how each of your solutions was obtained. A cryptarithm has not been solved until both of these conditions have been met.

Sometimes it may be of interest to consider a given cryptarithm in a system of numeration different from the one specified in the statement of the problem. Generalizations of a given problem are always welcomed.

The twenty problems which follow include several different types of cryptarithms. The problems are presented in the order in which they appeared in School Science and Mathematics - not necessarily in the order of their difficulty. For the first three of these problems the solutions are given immediately following the problem. These solutions may be used for instructional purposes. The remaining solutions are provided later in this work. The original solvers of the problems may be found in issues of School Science and Mathematics. They are omitted here to save space and to simplify the design.

CRYPTARITHM PROBLEMS

1. (2202, November 1950) Proposed by Alan Wayne.
In the following addition, each letter represents a digit with different letters representing different digits. Restore the digits.

$$\begin{array}{r} H\ E\ L\ P \\ +\quad T\ H\ E \\ \hline Y\ \emptyset\ U\ N\ G \end{array}$$

Solution:
We can see by inspection that H must be 9, Y must be 1 and Ø must be 0. Since H must be 9, N must be one less than L, and the sum of P and E must be less than 9. Now P and E can only be either 6 and 2, 5 and 3, 5 and 2, 4 and 3, 4 and 2, 3 and 2, or vice versa. Trying each of these, we see that the only ones that will work are 4 and 3. The rest of the letters then fall in and the result is as follows:

$$\begin{array}{r} 9364 \\ +\ 893 \\ \hline 10257 \end{array}$$

* *

2. (2263, June 1979) Proposed by Alan Wayne.
Find the digits represented uniquely by the letters in the addition

$$\begin{array}{r} \emptyset\ N\ C\ E \\ +\ C\ \emptyset\ D\ E\ D \\ \hline D\ I\ G\ I\ T\ S \end{array}$$

Solution:
Since the sum of two digits cannot exceed 18, we have immediately that D = 1, C = 9, and I = 0. Then proceeding from the right, the several columnar additions determine the relationships:

$$E + 1 = S$$
$$9 + E = T + 10, \text{ so } E - 1 = T$$
$$N + 1 + 1 = 10, \text{ so } N = 8$$
$$\emptyset + \emptyset + 1 = G + 10, \text{ so } G + 9 = 2\emptyset\ ,$$

hence G is odd. Then either G = 3, Ø = 6 or G = 5, Ø = 7. But the first two equations establish T, E, S as consecutive digits. Thus T = 2, E = 3, S = 4, G = 5, Ø = 7, and no letter represents 6. The reconstructed addition is:

7893 + 97131 = 105024

In any system of numeration with base r, we would have D = 1, C = r - 1, I = 0, N = r - 2, consecutive digits T, E, S, and G + r - 1 = 2Ø. When r = 9 there are no solutions, when r = 11 there are three solutions, and when r = 12 there are 5 solutions. The number of solutions increases with r, so the decimal scale is the only one in which the representations are unique. The solutions for r = 11 are:

0	1	2	3	4	5	6	7	8	9	X
I	D	G	T	E	S	Ø	-	-	N	C
I	D	T	E	S	-	G	-	Ø	N	C
I	D	-	T	E	S	G	-	Ø	N	C

* *

3. (2761, May 1961) Proposed by C. W. Trigg

TRITØNE is divisible by 3 and by 7. Furthermore T + RIT = ØNE. Find the digits uniquely represented by the letters in the decimal scale.

Solution:

From T + RIT = ØNE, it is immediately apparent that T + T = 10 + E, I + 1 = 10 + N, and R + 1 = Ø. Obviously I = 9, N = 0, 5 < T < 9, and R ≠ 8, 9. Since TRITØNE is divisible by 3, the sum of its digits is divisible by 3. Using the values given above the sum of the digits is readily found to be 20 + 2E + 2R. Using the only possible values for T: 6, 7, 8, we readily find E: 2, 4, 6, and R: 3, 1, 2 respectively. Checking the three possible restorations 6396402, 7197204, and 8298306 for divisibility by 7, we find that only the second meets this requirement and that the digits uniquely represented by the letters are T = 7, R = 1, I = 9, Ø = 2, N = 0, and E = 4.

* *

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4. (2801, February 1962) Proposed by Donald R. Byrkit.
In the following addition problem, each letter represents a unique digit. Find each digit

```
  F Ø R T Y
    T E N
    T E N
  ---------
  S I X T Y
```

* *

5. (3016, February 1966) Proposed by Charles W. Trigg.
In the following cryptarithm there is a one-to-one correspondence between the letters and the ten digits of the decimal system. The asterisks also represent digits. Reconstruct the division.

```
        A ) T I G E R ( A M I R
            T Ø
            ---
            * *
            U S
            ---
              * *
              * I
              ---
                * *
                Ø T
                ---
                  M
```

* *

6. (3201, June 1969) Proposed by Charles W. Trigg.
W. E. Akwit has been trying to perform arithmetic operations in the scale of eleven, but can't seem to get the knack, and

IS/HE = .MADMADMAD ···,

even though he realizes that MAD is a square way to be? In the sacle of his frustration, each letter of the cryptarithm uniquely represents a digit. Find its sole translation into a repeating decimal.

* *

7. (3207, October 1969) Proposed by Dale Woods.

FØUR + FØUR = SEVEN .

What digits are represented by each letter?

* *

8. (3284, February 1971) Proposed by Charles W. Trigg.

Before the days of the linotype, pied type was a problem. As the printer's devil said after he dropped a drawer full of type, "What a mess, it's always pi, pi, and more pi." If in the cryptarithm

ME/SS = .PIPIPI ···

the digits in the decimal system uniquely represented by the letters are in arithmetic progression, identify the fraction.

* *

9. (3424, June 1972) Proposed by Charles W. Trigg.

In the cryptarithm

AM + PM = DAY

the letters uniquely represent positive digits in the scale of six. Construct the addition.

* *

10. (3449, January 1973) Proposed by Frederic Schuler.

In the cryptarithm

ABC · ABC = DEFGHI

each different letter represents a unique digit. Find all possible solutions.

* *

11. (3476, June 1973) Proposed by Charles W. Trigg.

A puzzlist, accused of being despondent, replied, "I am not blue." Noting that there were no duplicated letters in his reply, he wondered if

(I)(AM)(NØT) = BLUE

was a valid cryptarithm in which each letter represented a different digit. What do you think?

* *

12. (3593, February 1976) Proposed by Alan Wayne.
Restore the following addition in which each letter represents precisely one decimal digit, and different letters represent different digits.

```
  A B C D E F G H I J
  A B C D E F G H I J
  -------------------
  B D F I A C E G H J
```

* *

13. (3607, May-June 1976) Proposed by Alan Wayne.
Solve the Arabic-Roman cryptarithmic system:

```
  TWØ + TWØ + TWØ = SIX
      VI +  VI = XII .
```

Each letter represents just one decimal digit, and different letters represent different digits.

* *

14. (3618, October 1976) Proposed by Charles W. Trigg.
In each of two different bases there is a solution of the doubly-true cryptarithm:

```
  Ø N E
  Ø N E
  -----
  T W Ø
```

in which the digits uniquely represented by the letters, in some order are consecutive. Find these two solutions and show that there are no others involving consecutive digits in any base.

* *

15. (3645, April 1977) Proposed by Alan Wayne.

How one potter answered another:

MY) JUG (IS
NØ
MUG
MUG

Suppose that each letter represents exactly one decimal digit, with different letters representing different digits, and that the entire pattern represents an arithmetical long division. Restore the digits.

* *

16. (3654, October 1977) Proposed by Alan Wayne.

How to make the slippery more slippery:

ED) ØIL (UP
AN
EEL
EEL

Regard the preceding as a long division problem involving positive integers in the decimal system of numeration. Each letter stands for just one digit and different letters stand for different digits. Restore the digits.

* *

17. (3673, January 1978) Proposed by Alan Wayne.

Solve the following two cryptarithms:

(a) A x DIVA = AVID

(b) I x SPØT = TØPS

* *

18. (3691, May-June 1978) Proposed by Alan Wayne.

This might be a possible Dutch treat:

M A D E
M E A D
E D A M

Interpret the above to be an addition problem in the decimal system, where each letter corresponds uniquely to a digit and conversely. Restore the digits.

* *

19. (3708, December 1978) Proposed by Alan Wayne.

Perhaps this might be a moral commentary:

```
  L I V E
  V I L E
---------
  E V I L
```

Regard the arrangement as an addition cryptarithm in which each letter represents uniquely a decimal digit, and different letters represent different digits. Restore the digits.

* *

20. (3739, November 1978) Proposed by Alan Wayne.

What might be said about De Moivre:

```
    H E
    I S
---------
  B I G
F Ø R
---------
T R I G
```

Regard the preceding pattern as an arithmetical multiplication of integers in the decimal system in which each digit has been replaced by one and only one letter, with different digits being replaced by different letters. Restore the digits.

ARITHMETICAL PASTIMES

Problem solving, in addition to being one of the most important aspects of mathematics, is a form of recreation for some people. Almost all of the following 25 arithmetical pastimes can be solved by using techniques which are found in first- or second-year high school algebra. Some of the mathematical topics it will be convenient to know about are the sum of an infinite geometric series, the sum of an arithmetic progression, the quadratic formula, different systems of numeration, the binomial theorem, and that the area of a triangle equals $\sqrt{s(s-a)(s-b)(s-c)}$ where a, b, and c are the measures of the sides of a triangle and s is its semi-perimeter. Of course, ingenuity and algebraic skill (and on occasions a hand calculator) will be helpful!

Following the initial three problems, solutions are given. The solutions to the remaining problems are provided later in this work. For solvers wishing assistance but not wishing a complete solution, hints and answers are provided separately from the solutions.

ARITHMETICAL PASTIMES

21. (467, June 1916) Proposed by F. L. Sanders.

The area of a triangle whose sides are 3, 4 and 5, is 6, and if the sides are 13, 14 and 15, the area is 84. Will someone find the next set, or general solution for such sets, of three consecutive integers which as the measures of the sides of a triangle give rise to an integral area?

Solution 1:

Let $2n$, $2n + 1$, $2n + 2$ be the sides of a triangle. Substituting for a, b, c, s in the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)} \qquad (1)$$

and simplifying, we get

$$A = \frac{2n+1}{4}\sqrt{3(2n-1)(2n+3)}$$

Since the product and square root of the product of odd numbers is odd, A can not be integral. Therefore, the smallest side of such a set of consecutive integers can not be even. Let $2n + 1$, $2n + 2$, $2n + 3$ be the sides. Substituting in (1) and simplifying,

$$A = (n+1)\sqrt{3n(n+2)}$$

A is integral when

$$3n(n+2) = a^2, \qquad (2)$$

a being any integer. From (2)

$$n = \frac{1}{3}\sqrt{3a^2 + 9} - 1$$

Choose a so that $3a^2 + 9$ is a perfect square and n is an integer, making A integral. In finding a only multiples of 3 need be considered, since a^2 must be divisible by 3. The values $a = 3$, $a = 12$, give 3, 4, 5 and 13, 14, 14 - two familiar sets. When $a = 45$, $n = 25$, and the sides are 51, 52, 53; when $a = 168$, $n = 96$ and we have 193, 194, 195. Multiples of 3 that make $3a^2 + 9$ a perfect square and n an integer may be found ad infinitum. To get the area, multiply $n + 1$ by a.

Solution 2:

Let $2x - 1$, $2x$, $2x + 1$ denote the sides of the triangle whose sides are consecutive whole numbers. The area of such a triangle will be

$$\sqrt{3x^2(x-1)(x+1)} = x\sqrt{3x^2-1)}$$

Now put

$$3(x^2-1) = 9y^2$$

Solving for x, we obtain

$$x = \sqrt{3y^2 + 1} \qquad (1)$$

The least value of y satisfying (1) is 0, which gives $x = 1$; but this value of x does not produce a triangle. The next value of y satisfying (1) is $y_1 = 1$, which gives $x_1 = 2$, and the sides of the first or least triangle are 3, 4, 5. The third value of y is $y_2 = 4$, which gives $x_2 = 7$, and the sides of the second triangle 13, 14, 15. The fourth value of y is $y_3 = 15$, which gives $x_3 = 26$, and the sides of the third triangle are 51, 52, 53; and so on. The successive values of x are easily seen to be connected by the relation

$$x_n = 4x_{n-1} - x_{n-2} \qquad (2)$$

and the sides of the n-th triangle are $2x_n - 1$, $2x_n$, $2x_n + 1$, where n is defined by (2).

From these expressions we find

4th,	193,	194,	195 .
5th,	723,	724,	725 .
6th,	2701,	2702,	2703 .
7th,	10083,	10084,	10085 .
8th,	37633,	37634,	37635 .
9th,	140451,	140452,	140453 .
10th,	524173,	524174,	524175 .
20th,	274758382273,	274758382274,	274758382275 .
30th,	144021200269567501,	144021200269567502,	144021200269567503 .

Further results and remarks by F. M. Phillips:

Let $y_n = 14y_{n-1} - y_{n-2}$

The areas of the first nine triangles are:

6

84

1170
16296
226974
3161340
44031786
613283664
8541939510

* *

22. (836, November 1924) Proposed by Richard Lewis.

Volume 1 of "Boys Mechanics" offers a very interesting system of doing multiplication on the fingers. The simplest system deals with numbers between 6 and 10. Let each thumb represent 6, the first finger 7, etc., the last finger representing 10. Then let the fingers representing the numbers to be multiplied touch, using one finger of each hand, the palms facing each other with "thumbs up". To find the desired product, add the number of fingers above and including the touching fingers times 10 to the product of the number of fingers below the touching fingers on one hand by the number of fingers below on the other hand. Prove that the method is correct.

Solution:

Let the number x be represented on one hand, and y on the other. Then x - 5 would be the number of fingers above and including the touching fingers on the hand, and y - 5 would be the number of fingers on the other. Also 10 - x is the number of fingers on one hand below the touching fingers, and 10 - y on the other hand. According to the rule

$$\begin{aligned} xy &= 10(x-5+y-5) + (10-x)(10-y) \\ &= 10x - 50 + 10y - 50 + 100 - 10y - 10x + xy \\ &= xy \end{aligned}$$

which proves the rule.

* *

23. (902, March 1926) Proposed by Michael Goldberg.

Show that the fourth power of the infinite continued fraction

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}$$

is the infinite continued fraction

$$7 - \cfrac{1}{7 - \cfrac{1}{7 + \cfrac{1}{7 - \ldots}}}$$

Solution:

Let x represent the first continued fraction and y represent the second continued fraction. Then

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}} = 1 + \frac{1}{x},$$

or,

$$x^2 - x - 1 = 0, \text{ and } x = \frac{1}{2} + \frac{\sqrt{5}}{2}, \text{ since clearly } x > 0.$$

$$x^4 = \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^4 = \frac{7}{2} + \frac{3\sqrt{5}}{2}.$$

Also,

$$y = 7 - \cfrac{1}{7 - \cfrac{1}{7 - \cfrac{1}{7 - \ldots}}} = 7 - \frac{1}{y}$$

or

$$y^2 - 7y + 1 = 0, \text{ and } y = \frac{7}{2} + \frac{3\sqrt{5}}{2}$$

Hence $y = x^4$.

* * * \

25. (1136, January 1931) Proposed by A. A. Shaw.
Prove that the product of any three consecutive integers is a multiple of 504 if the middle integer is a perfect cube.

* *

26. (1279, June 1933) From a university list of examination questions for H. S. pupils.
If the digit 7 is written at the right of a certain number that number is increased by 70,000. Find the number.

* *

27. (1286, November 1933) Proposed by Norman Anning.
The number N consists of p digits each of which is 6. Discover the nature of the product N(N+1). Give a proof which is valid when N is any positive integer.

* *

28. (1366, March 1935) Proposed by Norman Anning.
Show that the decimal fraction equivalent to the sum of the infinite series 1/10 + 2/100 + 3/1000 + ... does not contain the digit 8.

* *

29. (1521, February 1938) Proposed by Charles W. Trigg.
What is the smallest two-digit number which when expressed in another scale of notation merely has its digits reversed? What is the smallest three-digit number which behaves similarly?

* *

30. (1616, December 1939) Proposed by C. W. Trigg.
Find the two permutations of the ten digits which are multiples of 1901 and are also perfect squares. Show that they are the only ones.

* *

31. (1702, May 1941) Proposed by Paul C. Overstreet.
Show that for every perfect cube, n^3, there exists a series of n consecutive odd integers whose sum is n^3.

* *

32. (1961, June 1946) Proposed by Brother Felix John.
Find a simple value for

$$\frac{(4 + \sqrt{15})^{5/2} + (4 - \sqrt{15})^{5/2}}{(6 + \sqrt{35})^{5/2} - (6 - \sqrt{35})^{5/2}}$$

* *

33. (1979, October 1946) Proposed by J. F. Arena.
Find the value of the infinite product:

$$3^{1/3} \cdot 9^{1/9} \cdot 27^{1/27} \ldots$$

* *

34. (2040, October 1947) Proposed by Albert Hansen.
Prove that $x^2 - 3y = 17$ has no integral solution.

* *

35. (2095, December 1948) Proposed by Charles W. Trigg.
A nine-celled magic square is a 3 x 3 array in which the entries of each row, each column, and each diagonal have the same sum. Show that in a nine-celled magic square the central element must equal one-third of the constant. (The constant is the sum of the elements of any column, row, or diagonal.)

* *

36. (2105, January 1949) Proposed by Brother Felix John.
A man starts on a journey at 40 m.p.h. and reduces his speed by 2 miles every hour for an integral number of hours. If he had

maintained the original rate, the journey would have taken 2 3/4 hours less. What was the distance traveled?

* *

37. (2194, October 1950) Proposed by Alan Wayne.
What are the two ways in which fifty coins are the correct change for a dollar?

* *

38. (2701, April 1960) Proposed by Charles W. Trigg.
The consecutive odd numbers are grouped as follows:

1, (3,5); (7,9,11); (13,15,17,19);...

Find the sum of the numbers in the nth group.

* *

39. (2728, November 1960) Proposed by Cecil B. Read.
Find a two-digit number equal to the square of the tens' digit plus the square of the sum of its digits.

* *

40. (2945, November 1964) Proposed by Brother U. Alfred.
Determine the sum of all the positive proper fractions whose denominators are less than or equal to 100. (Note: Fractions need not be in their lowest terms. The series in question is

$$1/2 + 1/3 + 2/3 + 1/4 + 2/4 + 3/4 + 1/5 + \ldots)$$

* *

41. (3444, December 1972) Proposed by Cecil B. Read.
There are four whole numbers. Add 3 to the first, subtract 3 from the second, multiply the third by 3, divide the fourth by 3, and all are equal. If the difference between the two greater numbers is

36 more than the difference between the two smaller, what are the numbers?

* *

42. (3559, April 1975) Proposed by Charles W. Trigg.
The arithmetic mean of the twin primes 11 and 13 is the pentagonal number 12. Pentagonal numbers have the form $n(3n-1)/2$. Are there any other twin primes with a pentagonal mean?

* *

43. (3681, February 1978) Proposed by Joe Dan Austin.
Find numbers n(2), n(3), ..., n(11) so that for i = 2, 3, ..., 11 each n(i) (a) uses each of the digits 0, 1, 2, ..., 9 exactly once; (b) is divisible by i; and (c) is the largest number satisfying (a) and (b).

* *

44. (3699, October 1978) Proposed by Herta T. Freitag.
Let n be a given positive integer. If S_1 denotes the sum of the odd positive integers smaller than n and S_2 represents the analogous sum comprised of even positive integers, determine $|S_1-S_2|$.

* *

45. (3705, November 1978) Proposed by Alan Wayne.
Show that for each positive integer n there is an n-digit positive integer N such that N^2 starts with precisely n ones.

* *

ANSWERS TO CRYPTARITHM PROBLEMS

1.

```
  H E L P        9 3 6 4
    T H E          8 9 3
  -------        -------
  Y Ø U N G      1 0 2 5 7
```

2.

```
  Ø N C E        7 8 9 3
  C Ø D E D      9 7 1 3 1
  -------        -------
D I G I T S    1 0 5 0 2 4
```

3. T + RIT = ØNE 7 + 197 = 204

4.

```
  F Ø R T Y      2 9 7 8 6
    T E N          8 5 0
    T E N          8 5 0
  -------        -------
  S I X T Y      3 1 4 8 6
```

5.

```
A) T I G E R ( A M I R
   T Ø
   * *
   U S
     * *
     * 1
       * *
       Ø T
         M
```

```
9) 8 5 0 7 2 ( 9 4 5 2
   8 1
     4 0
     3 6
       4 7
       4 5
         2 2
         1 8
           4
```

6. IS/HE = .MADMADMAD... 18/64 = .2X92X92X9...

7. FØUR + FØUR = SEVEN 8635 + 8635 = 17270
8536 + 8536 = 17072

8. ME/SS = .PIPIPI... 19/33 = .575757...

9. AM + PM = DAY 34 + 54 = 132

10. ABC • ABC = DEFGHI 854 • 854 = 729316
807 • 807 = 651249
567 • 567 = 321489

11. (1)(AM)(NØT) = BLUE

(1)(26)(345) = 8970
(2)(14)(307) = 8596

12.

A B C D E F G H I J	1 2 3 4 5 6 7 8 9 0
A B C D E F G H I J	1 2 3 4 5 6 7 8 9 0
B D F I A C E G H J	2 4 6 9 1 3 5 7 8 0

13.

TWØ + TWØ + TWØ = SIX	267 + 267 + 267 = 801
VI + VI = XII	50 + 50 = 100

14.

Ø N E	base six: 2 3 4	base eight: 2 1 5
Ø N E	2 3 4	2 1 5
T W Ø	5 1 2	4 3 2

15.

MY) JUG (IS	14) 826 (59
NØ	70
MUG	126
MUG	126

16.

ED) ØIL (UP	16) 592 (37
AN	48
EEL	112
EEL	112

17.

(a) A x DIVA = AVID	(a) 9 x 1089 = 9801
(b) I x SPØT = TØPS	(b) 4 x 2178 = 8712

18.

M A D E
M E A D
E D A M

1 6 9 2
1 2 6 9
2 9 6 1

19.

L I V E
V I L E
E V I L

8 0 1 9
1 0 8 9
9 1 0 8

20.

H E	7 8
I S	5 2
B I G	1 5 6
F Ø R	3 9 0
T R I G	4 0 5 6

SOLUTIONS TO CRYPTARITHM PROBLEMS 4-20

4. (2801, February 1962) Proposed by Donald R. Byrkit.

In the following addition problem, each letter represents a unique digit. Find each digit.

```
  F Ø R T Y
      T E N
      T E N
  ---------
  S I X T Y
```

Solution:

This problem is the brain-child of Alan Wayne who proposed it as problem E751 in the American Mathematical Monthly in January 1947. Mr. Wayne has devised a number of these doubly-true addition cryptarithms. The following solution is essentially the one given by A. Chulick on page 413 of the August-September 1947 issue of the Monthly.

In the addition, we number the columns from right to left. From column 1 no digit is carried, so N = 0. Then E ≠ 0, so E = 5 and 1 is carried from column 2. Now F + 1 = S, so Ø = 9, I = 1, and 2 is carried from column 3. Hence, R + 2T + 1 = X + 20. If R = 8, T = 7, and X = 3, but F and S are consecutive integers, so it must be that R = 7, T = 8, and X = 4. Then F = 2, S = 3 and Y = 6. The unique reconstruction of the addition therefore is 29786 + 850 + 850 = 31486.

* *

5. (3016, February 1966) Proposed by C. W. Trigg.

In the following cryptarithm there is a one-to-one correspondence between the letters and the ten digits of the decimal system. The asterisks also represent digits. Reconstruct the division.

```
      A ) T I G E R ( A M I R
          T Ø
          * *
          U S
          * *
          * I
            * *
            Ø T
            M
```

Solution:

From the first, third, and fourth divisions: A • A = T Ø, A • I = * I, and A • R = Ø T and the only digit replacements which satisfy all these conditions are: A = 9, T = 8, Ø = 1, R = 2, and I = 5. Making these substitutions leads to the conclusions that M • 9 = 36, and M = 4, U = 3, S = 6, and therefore E must equal 7 and G zero. The total reconstruction is:

```
      9 ) 8 5 0 7 2 ( 9 4 5 2
          8 1
          ---
            4 0
            3 6
            ---
              4 7
              4 5
              ---
                2 2
                1 8
                ---
                  4
```

* * * \

Since MAD is a square, its square root $\leq 33 = \lfloor\sqrt{XXX}\rfloor$. Therefore, HE > IS = 13, 32, or 18 times a square factor. Thus there are four possibilities;

13/87 = .169169...

51/87 = .653653...

32/35 = .X14X14...

18/64 = .2X92X9...

The first three are eliminated because of duplicate digits, so the last one is the unique solution. If the restriction of being a square is removed from MAD there are fifteen other solutions:

$42/87 = .\overline{536}$,	$24/87 = .\overline{301}$,	$25/87 = .\overline{314}$,	$41/87 = .\overline{523}$,
$19/87 = .\overline{235}$,	$29/87 = .\overline{365}$,	$32/87 = .\overline{406}$,	$2X/64 = .\overline{503}$,
$18/32 = .\overline{5X7}$,	$1X/35 = .\overline{609}$,	$49/87 = .\overline{615}$,	$54/87 = .\overline{691}$,
$29/35 = .\overline{8X7}$,	$53/64 = .\overline{912}$,	$57/64 = .\overline{981}$.	

The first of these is interesting in that the digits involved are consecutive. In the next three, the digits of IS and MAD are consecutive. Nine is the only other number base less than thirteen in which this cryptarithm has a solution.

* *

7. (3207, October 1969) Proposed by Dale Woods.

FØUR + FØUR = SEVEN

What digits are represented by each letter?

Solution:

It is easily seen that S = 1, F $\leq$ 5, R $\neq$ 0, and Ø $\neq$ 0.

Suppose F = 5. If 2Ø < 10 then E = 0. It follows that U is 0.
But E and U must be different.
If 2Ø $\geq$ 10 then E = 1. This conflicts with the fact that S = 1.

Thus F $\neq$ 5.

Suppose F = 6. If 2Ø < 10 then E = 2. But then U must equal 1.
However S = 1.
If 2Ø $\geq$ 10 then E = 3, and U = 1 - an impossibility.

Thus F $\neq$ 6.

Suppose F = 7. If 2Ø < 10 then E = 4, and U = 2. Because both 2R and 2Ø must be less than 10, we must have R = Ø = 3. This cannot be the case.

If $2\emptyset \geq 10$ then E = 5, and U = 2. The latter requirement makes $\emptyset \neq 6$ and $R \neq 6$. If $\emptyset = 8$ then R = 9 so $N = 8 = \emptyset$. If $\emptyset = 9$ then $R = 8 = V$.

Thus $F \neq 7$.

Suppose F = 9. If $2\emptyset < 10$ then E = 8, and U = 4. Since both 2R and $2\emptyset$ must be less than 10, (R = 2 or R = 3) and ($\emptyset$ = 2 or $\emptyset$ = 3). It follows that either $R = \emptyset$ or $V = 4 = U$ or $N = 4 = U$. If $2\emptyset \geq 10$ then E = 9. But F = 9.

Thus $F \neq 9$.

If F = 8 and $2\emptyset < 10$ then E = 6 and U = 3. Because both 2R and $2\emptyset$ must be less than 10, ($\emptyset$ = 2 or $\emptyset$ = 4) and (R = 2 and R = 4). This yields $R = \emptyset$ or $V = 4 = R$ or $N = 4 = \emptyset$.

If F = 8 and $2\emptyset \geq 10$ then E = 7 and U = 3. The only possibilities for $\emptyset$ and R are 5, 6, and 9. Nine is easily eliminated. The remaining values for $\emptyset$ and R give the following two solutions:

8536	8635
8536	8635
17072	17270

* *

8. (3284, February 1971) Proposed by C. W. Trigg.

Before the days of the linotype, pied type was a problem. As the printer's devil said after he dropped a drawer full of type, "What a mess, it's always pi, pi, and more pi." If in the cryptarithm

ME/SS = .PIPIPI ...

the digits in the decimal system uniquely represented by the letters are in arithmetic progression, identify the fraction.

Solution:

Let	F = .PIPIPI ...
then	100F = PI.PIPIPI ...
so	99F = PI
Thus	ME/SS = PI/99

Now since ME $\neq$ PI, S $\neq$ 9, and since M < S, S $\neq$ 1, Hence, S = 3, PI = 3(ME), and 13 < ME < 30. There are six pairs (ME,PI) within this range in which no duplicate digits nor 3 appear:

(16,48), (18,54), (19,57), (26,78), (27,81), and (29,87). The only pair into which 3 fits to form a digital arithmetic progression leads to the unique solution

$$19/33 = .575757\ldots .$$

* *

9. (3424, June 1972) Proposed by C. W. Trigg.

In the cryptarithm AM + PM = DAY, the letters uniquely represent positive digits in the scale of six. Construct the addition.

Solution:

Clearly D = 1.

Also A + P + 1 = 6 + A so P = 5.

Since M + M $\geq$ 6, M = 3 or M = 4.

Therefore PM = 53 or 54.

PM = 53 gives AM = 23 or 43, so DAY = 120 or 140.

PM = 54 gives AM = 34, so DAY = 132.

Since the digits must be positive, the unique solution is

$$34 + 54 = 132$$

* *

10. (3449, January 1973) Proposed by Frederic Schuler.

In the cryptarithm

$$ABC \cdot ABC = DEFGHI$$

each different letter represents a unique digit. Find all possible solutions.

Solution 1:

Clearly C $\neq$ 0, 1, 5, 6. If C = 2, we readily find B $\neq$ 0,1,2,3,4,6,8. With a desk calculator or set of tables we can easily test for A. Similarly if C = 3, B $\neq$ 0,2,3,4,6,8,9, etc. Obviously all numbers containing repeated digits can be eliminated immediately. Thus we find

$$854 \cdot 854 = 729316$$
$$807 \cdot 807 = 651249$$
$$567 \cdot 567 = 321489$$

Solution 2:

Since $10^5 < ABC \cdot ABC < 10^6$, $317 \leq ABC \leq 999$. A $\neq$ D further

reduces this to $317 \leq ABC \leq 948$.

```
10    REM   THIS BASIC LANGUAGE PROGRAM WILL FIND ALL SOLUTIONS
            OF THE CRYPTARITHM ABC*ABC=DEFGHI.
20    REM   THE VARIABLES A(1) THRU A(9) REPRESENT THE DISTINCT
            DIGITS A THRU I.
30    FOR N=317 TO 948
40    LET Y=N*N
50    LET A(1)=INT(N/100)
60    LET A(2)=INT((N-100*A(1))/10)
70    LET A(3)=N-100*A(1)-10*A(2)
80    LET A(4)=INT(Y/100000)
90    LET B4=Y-100000*A(4)
100   LET A(5)=INT(B4/10000)
110   LET B5=B4-10000*A(5)
120   LET A(6)=INT(B5/1000)
130   LET B6=B5-1000*A(6)
140   LET A(7)=INT(B6/100)
150   LET B7=B6-100*A(7)
160   LET A(8)=INT(B7/10)
170   LET A(9)=B7-10*A(8)
180   FOR I=1 TO 8
190   FOR J=I+1 TO 9
200   IF A(I)=A(J)
      THEN 240
210   NEXT J
220   NEXT I
230   PRINT N;Y
240   NEXT N
250   END
RUN
 567   321489
 807   651249
 854   729316
```

Comment by C. W. Trigg: The answers $(854)^2 = 729{,}316$ and $(567)^2 = 321{,}489$ are the answers to Problem E89, American Mathematical Monthly, 41 (November 1934), 579, namely: "Find two integers, one the square of the other, which together contain each of the nine digits from one to nine just once."

The doubly true cryptarithm, $(TWØ)^2$ = SQUARE has the same solutions.

Other encryptments with the same solutions are:

$(WØN)^2$ = PRIZES, $(CØW)^2$ = MILKER, $(TRE)^2$ = MUCHAS, WØMEN'S = $(LIB)^2$, (BAD)(BAD) = PØLICY, GRAPES = $(VIN)^2$, $(BUG)^2$ = INSECT, AND $(DAY)^2$ = NIGHTS.

* *

11. (3476, June 1973) Proposed by C. W. Trigg.

A puzzlist, accused of being despondent, replied, "I am not blue." Noting that there were no duplicated letters in his reply, he

wondered if

$$(I)(AM)(N\emptyset T) = BLUE$$

was a valid cryptarithm in which each letter represented a different digit. What do you think?

Solution:

In $(I)(AM)(N\emptyset T) = BLUE$, only $\emptyset$, L, U, or E can be zero, and $(I)(A)(N) < 10$.

If $\{I, A, N\} = \{1, 2, 4\}$ and $\emptyset \neq 0$, the smallest possible product is $1(23)(456) = 10488$. For $\emptyset = 0$, since $2(13)(405) = 10530$, $I = 1$. Whether $(A, N) = (2, 4)$ or $(4, 2)$, $M = 3$, and there are no solutions.

If $\{I, A, N\} = \{1, 2\ 3\}$, there are two solutions, $1(26)(345) = 8970$ and $2(14)(307) = 8596$.

* *

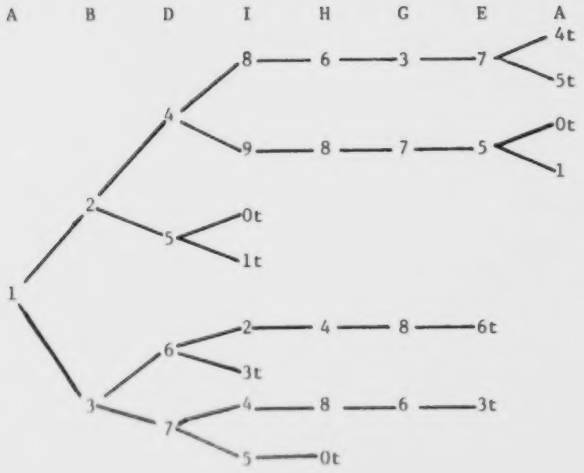
12. (3593, February 1976) Proposed by Alan Wayne.

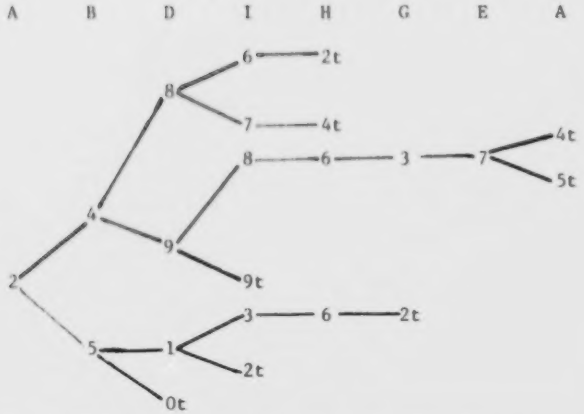
Restore the following addition in which each letter represents precisely one decimal digit, and different letters represent different digits.

$$\begin{array}{r} A\ B\ C\ D\ E\ F\ G\ H\ I\ J \\ \underline{A\ B\ C\ D\ E\ F\ G\ H\ I\ J} \\ B\ D\ F\ I\ A\ C\ E\ G\ H\ J \end{array}$$

Solution:

It is immediately obvious that $J = 0$ and $A < 5$. In the following two tree arrays, t indicates that a branch has terminated in a value which is not possible.





Similar arrays for A = 3 and A = 4 yield no new solutions.

Hence we have the unique restoration:

1 2 3 4 5 6 7 8 9 0

1 2 3 4 5 6 7 8 9 0

2 4 6 9 1 3 5 7 8 0

* *

13. (3607, May-June 1976) Proposed by Alan Wayne.

Solve the Arabic-Roman cryptarithmic system:

$$TW\emptyset + TW\emptyset + TW\emptyset = SIX$$
$$VI + VI = XII$$

Each letter represents just one decimal digit, and different letters represent different digits.

Solution:

From the second equation it is immediate that I = 0, so V = 5 and X = 1. In the first equation, $\emptyset = 7$ is the only value which gives 1 as units' digit. Also 3W + 2 has units' digit 0 so W = 6. Because $3T + 2 \leq 9$, T = 2. Therefore the unique solution is:

$$267 + 267 + 267 = 801$$
$$50 + 50 = 100$$

* *

14. (3618, October 1976) Proposed by C. W. Trigg.

In each of two different bases there is a solution of the doubly-true cryptarithm:

$$\begin{array}{r} \emptyset\ N\ E \\ \underline{\emptyset\ N\ E} \\ T\ W\ \emptyset \end{array}$$

in which the digits uniquely represented by letters, in some order are consecutive. Find these two solutions and show that there are no others involving consecutive digits in any base.

Solution:

The five different letters, N, Ø, T, W, E, each represent a distinct digit. The following discussion is predicated upon these five digits, in some order, being consecutive in a base $b \geq 5$. Since zero cannot properly begin an integer, $\emptyset \neq 0$ so $E \neq 0$. If $\emptyset > 4$, then $T > \emptyset + 4$, so the span of Ø to T, inclusive, will be greater than five in any eligible base. Hence, $\emptyset \leq 4$. It follows that, in even bases, E = 1, 2, (b/2 + 1), or (b/2 + 2). In odd bases, E = 1, 2, (b+1)/2, or (b+3)/2.

(A) In any base, if E = 1, Ø = 2, and T = 4, then (N,W) = (0,3), (3,0), (3,5) or (5,3). In no case is the second of the pair equal to twice the first.

In any base, if E = 1, Ø = 2, and T = 5, then (N,W) = (3,4) or

(4,3). Neither fits the addition pattern.

(B) If E = 2, Ø = 4, and T ≥ 8, then the five digits must be 4, 5, 6, 7, and 8. Since E does not fall within this span, the situation is impossible.

(C) If N = 0, W = 1, and the consecutive digits are 0,1,2,3,4, then it must be that T = 4, Ø = 2, and E = 3. But this would require that b = four, which base does not contain a digit 4. Hence N ≠ 0.

(D) If E = (b + 1)/2 in an odd base, then Ø = 1 and T = 2 or 3. It follows that b = 5, 7, or 9, since E will fall outside the 1 to 5 span for b ≥ 11.

If b = five, E = 3 so T = 2, but (N,W) = (4,0) does not fit the addition pattern.

If b = seven, E = 4, but none of (N,W) = (3,0), (3,5),(5,3),(2,0), (2,5),(5,2) form a proper addition.

If b = nine, E = 5, T = 2, but neither of (N,W) = (3,4) or (4,3) provides a solution.

(E) If E = (b+3)/2 in an odd base, then Ø = 3 and T = 6 or 7, so b > 5. To provide for five consecutive digits, b < 13.

If b = seven, E = 5, and T = 6. But neither (2,4) nor (4,2) satisfy (N,W).

If b = nine, then E = 6 and T = 7. But (N,W) cannot be either (4,5) or (5,4).

If b = eleven, E = 7 and T = 6. Again, (N,W) cannot be either (4,5) or (5,4).

(F) If E = b/2 + 2 in an even base, then Ø = 4 and T = 8. Consequently, the five consecutive digits are 4, 5, 6, 7, 8.

If b = ten, E = 7, and 2N + 1 = W. This is not satisfied by either (5,6) or 6,5).

If b = twelve, E = 8 = T.

If b ≥ fourteen, E > 8, so it does not fall within the required span.

(G) If E = b/2 + 1 in an even base, then Ø = 2, and T = 4 with 2N + 1 = W, or T = 5 with 2N + 1 = W + b.

If b = six, E = 4, Ø = 2, and T = 5 with 2N = W + 5, so N = 3, W = 1, and the solution is:

$$\begin{array}{r} 2\ 3\ 4 \\ +\ 2\ 3\ 4 \\ \hline 5\ 1\ 2 \end{array}$$

If b = eight, E = 5, Ø = 2, and T = 4 with 2N + 1 = W. Then (N,W) = (1,3),(3,1),(3,6), or (6,3). These yield the single solution:

$$\begin{array}{r} 2\,1\,5 \\ \underline{2\,1\,5} \\ 4\,3\,2 \end{array}$$

If b = ten, E = 6, Ø = 2, the T = 4 and 2N + 1 = W, with (N,W) = (3,5) or (5,3); or T = 5 and 2N = W + 9 with (N,W) = (3,4) or (4,3). None of these leads to a solution.

If b > ten, E > 6. But Ø = 2, so no sequence of five consecutive digits is possible.

Consequently, there are only two solutions (those shown above) of the cryptarithm involving consecutive digits. Note the remarkable curiosity that between these solutions the digits of ØNE and TWØ are interchanged and reversed.

* *

15. (3645, April 1977) Proposed by Alan Wayne.

How one potter answered another:

$$\begin{array}{r} \text{MY)}\ \text{JUG}\ \text{(IS} \\ \underline{\text{NØ}\phantom{\text{G}}} \\ \text{MUG} \\ \underline{\text{MUG}} \end{array}$$

Suppose that each letter represents exactly one decimal digit, with different letters representing different digits, and that the entire pattern represents an arithmetical long division. Restore the digits.

Solution:

The given problem may be solved by finding the simultaneous solution of the equations:

$$\text{(a)}\ \begin{array}{r} \text{MY} \\ \underline{\times\ \text{I}} \\ \text{NØ} \end{array} \qquad \text{(b)}\ \begin{array}{r} \text{MY} \\ \underline{\times\ \text{S}} \\ \text{MUG} \end{array} \qquad \text{(c)}\ \begin{array}{r} \text{JUG} \\ \underline{-\text{NØ}} \\ \text{MUG} \end{array}$$

It follows from (c) than the letter Ø must be zero and that J - N = M. From (a) either (Y = 5 and I is 2, 4, 6, or 8) or (I = 5 and Y is 2, 4, 6, or 8).

Since Y, Ø, and G are distinct, (b) tells us that Y ≠ 5. Hence I = 5 and Y = 2, 4, 6, or 8.

Replacing I by 5 in (a) we see that M = 1, so J = N + 1. If Y = 2, we have

12	12	JUG
x 5	x S	-60
60	1UG	1UG

The middle equation forces S to be 9, yielding U = 0. But this contradicts the condition that different letters must represent different digits.

In a similar manner the cases Y = 6 and Y = 8 can be eliminated. Thus Y = 4 and our equations become

14	14	8UG
x 5	x S	-70
70	1UG	1UG

Because S ≠ 8, the second equation makes S = 9 and hence G = 6 and U = 2. The restored division is

```
14) 826 (59
    70
    126
    126
```

* *

16. (3654, October 1977) Proposed by Alan Wayne.

How to make the slippery more slippery:

```
ED) ØIL (UP
    AN
    EEL
    EEL
```

Regard the preceding as a long division problem involving positive integers in the decimal system of numeration. Each letter stands for just one digit and different letters stand for different digits. Restore the digits.

Solution:

The given problem may be solved by finding the simultaneous solution of the equations:

(a)	(b)	(c)
ED	ØI	ED
x U	-AN	x P
AN	EE	EEL

It follows from (a) that U $\geq$ 2 and E = 1,2,3, or 4. If E = 4 then 4D x P = 44L. But 48 x 9 = 432 so E $\neq$ 4. If E = 3 then 3D x P = 33L. But 38 x 9 = 342 and 37 x 9 = 333, so E $\neq$ 3. If E = 2 then 2D x P = 22L. The only products of this form are: 25 x 9 = 225 and 28 x 8 = 224. Each results in a duplication of digits. Hence E = 1.

Since 1D x P = 11L and 1D x U = AN, U < P and P = 6,7,8, or 9. The only possibilities for 1D x P = 11L are: 13 x 9 = 117, 14 x 8 = 112, 16 x 7 = 112, and 9 x 6 = 114.

Now the following table of admissible values can be constructed:

E	P	D	L	U	A	N	I	Ø
1	6	9	4	2	3	8		
				3	5	7	8	
	7	6	2	3	4	8	9	5
	8	4	2					
	9	3	7	4	5	2		

yielding the unique solution

```
      16) 592 (37
          48
          112
          112
```

* *

17. (3673, January 1978) Proposed by Alan Wayne.

Solve the following two cryptarithms:

(a) A x DIVA = AVID
(b) I x SPØT = TØPS

Solution to (a):

Clearly A $\neq$ 0 and A $\neq$ 1. Thus AVID > DIVA, so A > D. The units' digit of A · A is D and A · D $\leq$ A $\leq$ 9. A = 9 and D = 1 are the values of A and D which satisfy the above requirements. Hence 9(1000 + 100I + 10V + 9) = 9000 + 100V + 10I + 1 so 89I = V - 8. Therefore I = 0 and V = 8. This gives the unique solution 9 x 1089 = 9801.

Solution to (b):

Clearly I $\neq$ 0 and I $\neq$ 1, so I $\geq$ 2. Also S · I $\leq$ T $\leq$ 9. Thus

$S \leq 4$. If $S = 4$ then $I = 2$ and ($T = 8$ or $T = 9$). Since $2 \cdot T$ must have units' digit 4 neither $T = 8$ nor $T = 9$ works. Hence $S \neq 4$. If $S = 3$ then $I = 2$ and $6 \leq T \leq 9$. But $2 \cdot T$ must have units' digit 3 so none of the possibilities for T works. If $S = 2$ then ($I = 3$ or $I = 4$) and $6 \leq T \leq 9$. Because $I \cdot T$ must have units' digit 2, all of the possibilities except $I = 4$ and $T = 8$ can be eliminated. With $S = 2$, $I = 4$, and $T = 8$, the cryptarithm becomes $4 \times 2P\emptyset 8 = 8\emptyset P2$. Since there is no "carry" involved in obtaining the digit 8 in the product, $4P \leq 9$ so ($P = 0$ or $P = 1$). Next we observe that $4 \cdot \emptyset + 3 \equiv P \pmod{10}$, which makes P odd. Thus $P \neq 0$. This makes $P = 1$ and $\emptyset = 7$, yielding the solution $4 \times 2178 = 8712$. If $S = 1$, the only possibilities for T and I are 7 and 3 respectively because the units' digit of $I \cdot T$ is 1. This gives us $3 \times 1P\emptyset 7 = 7\emptyset P1$. But $3 \times 19\emptyset 7 < 3 \times 2000 = 6000 < 7\emptyset P1$. Hence $S \neq 1$. Therefore the cryptarithm has the unique solution $4 \times 2178 = 8712$.

Charles W. Trigg reports:

On page 44 of A Mathematician's Apology, Cambridge, 1940, the statement is made by G. H. Hardy that "8712 and 9801 are the only four-figure numbers which are integral multiples of their 'reversals': $8712 = 4(2178)$ and $9801 = 9(1089)$." These are also given in W. W. Rouse-Ball, Mathematical Recreations and Essays, Macmillan, 1956, p. 13.

Thus (a) $A(DIVA) = AVID \rightarrow 9(1089) = 9801$; and

(b) $I(SP\emptyset T) = T\emptyset PS \rightarrow 4(2178) = 8712$.

In "Reversible Multiples," Journal of Recreational Mathematics 8(2), 1975-76, pp. 89-91, G. A. Grimm and David W. Ballew list all numbers (base ten) less than 10^9 that are multiples of the reversal of their digits.

Irwin K. Feinstein notes the following pattern related to solutions of (a):

base 9	1078	base 8	1067	base 7	1056
	x 8		x 7		x 6
	8701		7601		6501
base 6	1045	base 5	1034	base 4	1023
	x 5		x 4		x 3
	5401		4301		3201

* *

18. (3691, May-June 1978) Proposed by Alan Wayne.

This might be a possible Dutch treat:

```
  M A D E
  M E A D
---------
  E D A M
```

Interpret the above to be an addition problem in the decimal system, where each letter corresponds uniquely to a digit and conversely. Restore the digits.

Solution 1:

It follows from consideration of the tens' column than D = 0 or D = 9. If D = 0 then E = M which is impossible. Hence D = 9. From the ones' column E + 9 = 10 + M so E = 1 + M. By looking at the thousands' column we see that 1 + M + M = E or M + M = E. Substituting 1 + M for E in each of these equations yields M = 0 or M = 1. But M ≠ 0 so M = 1 and E = 2. We now have

```
  1  A  9  2
  1  2  A  9
-------------
  2  9  A  1
```

Thus 1 + A + 2 = 9 so A = 6. The unique solution is:

```
  1  6  9  2
  1  2  6  9
-------------
  2  9  6  1
```

Solution 2:

The following is a BASIC language program to solve this program:

```
10   FOR M=1 TO 9
20   FOR A=0 TO 9
30   IF A=M THEN 170
40   FOR D=0 TO 9
50   IF D=M OR D=A THEN 160
60   FOR E=1 TO 9
70   IF E=M OR E=A OR E=D THEN 150
80   X=(1000*M)+(100*A)+(10*D)+E
90   Y=(1000*M)+(100*E)+(10*A)+D
100  Z=(1000*E)+(100*D)+(10*A)+M
110  IF X+Y<>Z THEN 150
120  PRINT "MADE ="; M;A;D;E
130  PRINT "MEAD ="; M;E;A;D
140  PRINT "EDAM ="; E;D;A;M
150  NEXT E
160  NEXT D
170  NEXT A
180  NEXT M
190  END
RUN
MADE =  1  6  9  2
MEAD =  1  2  6  9
EDAM =  2  9  6  1
DONE
```

* *

19. (3708, December 1978) Proposed by Alan Wayne.

Perhaps this might be a moral commentary:

$$\begin{array}{r} L\ I\ V\ E \\ \underline{V\ I\ L\ E} \\ E\ V\ I\ L \end{array}$$

Regard the arrangement as an addition cryptarithm in which each letter represents uniquely a decimal digit, and different letters represent different digits. Restore the digits.

Solution:

Consider the more general case in base $b \geq 4$. No more than 1 can be carried over from the addition of a digit pair. From the b^3 column we see that $L < E$. Hence, from the units' column, $E + E = b + L$ so $1 + V + L = I$ or $1 + V + L = b + I$. Furthermore, because $E \neq 1$, there can be no carry from the b^2 column to the b^3 column. Thus $L + V = E$ so $1 + E = I$ or ($1 + E = b + I$ and $I = 0$). The latter is the only solution compatible with no carry from the b^2 column. Therefore, $L = b - 2$, $V = 1$, and the general reconstruction of the addition is

$$\begin{array}{cccc} b-2 & 0 & 1 & b-1 \\ \underline{1} & \underline{0} & \underline{b-2} & \underline{b-1} \\ b-1 & 1 & 0 & b-2 \end{array}$$

Some representative specific reconstructions are:

Base four	Base ten	Base twelve
2013	8019	A01B
1023	1089	10AB
3102	9108	B10A

* *

20. (3739, November 1978) Proposed by Alan Wayne.

What might be said about De Moivre:

$$\begin{array}{r} H\ E \\ \underline{I\ S} \\ B\ I\ G \\ \underline{F\ \emptyset\ R} \\ T\ R\ I\ G \end{array}$$

Regard the preceding pattern as an arithmetical multiplication of integers in the decimal system in which each digit has been

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replaced by one and only one letter, with different digits being replaced by different letters. Restore the digits.

Solution:

Obviously, R = 0. Since E cannot be 0 or 5, and I · E has units' digit 0, I = 5 and E is even. Also F is 1,2,3, or 4. Suppose F = 4. Then T = 5. Thus F ≠ 4.

Suppose F = 2. Then T = 3, and H = 4. Since E is even, E = 6 or 8; but these values lead to Ø = 3 or 4 respectively. Hence F ≠ 2.

Suppose F = 1. Then T = 2, and H = 3. But then B = 1 or 2, which is impossible. Hence F ≠ 1.

Therfore F = 3, whence T = 4. Then H = 6 or 7. If H = 6, then (E,G) = (2, 8) or (8, 2), and as a result in either case, S = 4, a contradiction. Hence H = 7.

Since B + Ø = 10, then (B, Ø) = (2, 8) or (1, 9).

The first of these is impossible because E and G are both even.

Hence B = 1, and Ø = 9, whence E = 8, S = 2, and G = 6.

The solution is unique, namely:

$$\begin{array}{r} 78 \\ \times\ 52 \\ \hline 156 \\ 390\ \ \\ \hline 4056 \end{array}$$

* *

ARITHMETICAL PASTIMES - HINTS AND ANSWERS

21. The next set is 51, 52, 53. This triangle has 1170 as its area.

22. HINT: Let x be the number represented on one hand. Then x - 5 is the number of fingers above and including the touching finger on this hand. Also 10 - x is the number of fingers below the touching finger on this hand.

23. HINT:

$$\text{Let } x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}} \qquad \text{Note that } x = 1 + \frac{1}{x}.$$

24. 998001
494209

25. HINT: Show that the product is divisible by 7, 8, and 9, and hence by $7 \cdot 8 \cdot 9 = 504$.

26. 7,777

27. HINT: Try some examples.

28. $S = 10/81 = 0.\overline{1234567890}$

29. $21_{\text{three}} = 12_{\text{five}}$

$211_{\text{three}} = 112_{\text{four}}$

30. $45624^2 = 2081549376$

$91248^2 = 8326197504$

31. HINT: Try several examples. Consider the cases n even and n odd.

32. HINT:
$$4 + \sqrt{15} = \left[\frac{\sqrt{5} + \sqrt{3}}{\sqrt{2}}\right]^2$$

33. HINT:
$$9^{1/9} = 3^{2/9} \text{ and } \frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \ldots = \frac{3}{4}$$

34. HINT: Consider the cases $x = 3n$, $x = 3n + 1$, and $x = 3n = 1$. Check for divisibility by 3.

35. HINT: Consider the four sums which involve the central element of the magic square.

36. 330 miles

37. 45 pennies, 2 nickels, 2 dimes and
1 quarter, 40 pennies, 8 nickels, and 2 dimes

38. HINT: $3 + 5 = 8 = 2^3$

39. 50

41

40. 2,475

41. $w = 24$, $x = 30$, $y = 9$, and $z = 81$.

42. HINT: Twin primes p and p +2, where $p > 3$, have the forms $6k - 1$ and $6k + 1$.

43.

i	n(i)
2	9,876,543,210
3	9,876,543,210
4	9,876,543,120
5	9,876,543,210
6	9,876,543,210
7	9,876,543,201
8	9,876,543,120
9	9,876,543,210
10	9,876,543,210
11	9,876,524,130

44. $|S_1 - S_2| = \lceil n/2 \rceil$ where $\lceil m \rceil$ denotes the greatest integer not exceeding m.

45. HINT: $4^2 = 16$ and $34^2 = 1156$.

ARITHMETICAL PASTIMES - SOLUTIONS 24 - 45

24. (1072, October 1929) Proposed by E. de la Garza.

9 is the square root of 81, and 8 + 1 = 9; 45 is the square root of 2025 and 20 + 25 = 45; 55 is the square root of 3025, and 30 + 25 = 55. Find a number of six figures satisfying the same condition.

Solution:

Let x represent the left three digits, and y the right three digits. Then $1000x + y = (x+y)^2$, where x and y are less than 1000. Let $(x+y) = z$. Then

$$z^2 = 1000x + y = z + 999x$$

Hence $z(z-1) = 999x = (27)(37)x$. Since z and z-1 are relatively prime, we have one of the following:

(1) $z = 99m$
$z-1 = n$
whence $999m = n + 1$
The smallest non-negative integral values of m and n which satisfy the above equation are m = 1 and n = 998.
Thus $z^2 = 999^2 = 998001 = (998 + 001)^2$.

(2) $z = 37m$
$z-1 = 27n$
whence $37m = 27n + 1$.
The smallest non-negative integral values of m and n which satisfy the above equation are m = 19 and n = 26.
Thus $z^2 = 703^2 = 494209 = (494 + 209)^2$.

(3) $z = 27m$
$z-1 = 37n$
whence $27m = 37n + 1$.
The smallest non-negative integral values of m and n which satisfy the above equation are m = 11 and n = 8.
Thus $z^2 = 297^2 = 88209 = (88 + 209)^2$. However, 88209 is not a six-digit number.

(4) $z = m$
$z-1 = 999n$
whence $m = 999n + 1$.
The smallest non-negative integral values of m an n which satisfy the above equation are m = 1 and n = 0.
Thus $z^2 = 1^2 = 1 = (000+001)^2$.
However, 1 is not a six-digit number.

* *

25. (1136, January 1931) Proposed by A. A. Shaw.

Prove that the product of any three consecutive integers is a multiple of 504 if the middle integer is a perfect cube.

Solution:

Our product $(n^3 - 1)\ n^3\ (n^3 + 1)$ equals $(n-1)(n^2+n+1)\ n^3\ (n+1)(n^2-n+1)$. We shall show that at least one of these factors must be divisible by 7, one by 8 and one by 9, whose product is 504. If x be any positive integer, all values of n are included below.

$n = 7x$.

$n = 7x+1$, whence $n-1 = 7x$.

$n = 7x+2$, whence $n^2+n+1 = 49^2+35x+7$.

$n = 7x+3$, whence $n^2-n+1 = 49^2+35x+7$.

$n = 7x+4$, whence $n^2+n+1 = 49x^2+63x+21$.

$n = 7x+5$, whence $n^2-n+1 = 49x^2+63x+21$.

$n = 7x+6$, whence $n+1 = 7x+7$. Thus in every case 7 is a factor of the product.

Again, $n = 2x$, whence $n^3 = 8x^3$.

or $n = 2x+1$, whence $n-1$ and $n+1$ are consecutive even numbers, so their product is divisible by 8.

Finally, $n = 3x$, whence $n^3 = 27x^3$.

$n = 3x+1$, whence $n^3-1 = 27x^3+27^2+9x$.

$n = 3x+2$, whence $n^3+1 = 27x^3+27^3+54x^2+36x+9$.

Thus the product is also divisible by 9.

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26. (1279, June, 1933) From a university list of examination questions for H. S. pupils.

If the digit 7 is written at the right of a certain number that number is increased by 70,000. Find the number.

Solution:

Let x equal the number which is to be found.

Then $10x + 7 = x + 70,000$

so $9x = 69,993$.

Thus the required number $x = 7,777$.

* *

27. (1286, November 1933) Proposed by Norman Anning.

The number N consists of p digits each of which is 6. Discover

the nature of the product N(N+1). Give a proof which is valid when N is any positive integer.

Solution:

We notice that 6 x 7 = 42, 66 x 67 = 4422, etc. In general, N(N+1) is an integer of 2p digits. Each of its first p digits is 4 and each of the last p digits is 2.

To prove this we use the following notation, for convenience and compactness.

Let $a_0 = 1 + 10 + 10^2 + \ldots + 10^{p-1}$

$a_1 = 10 + 10^2 + \ldots + 10^p$, etc.

Then we may write $N = 6(10^{p-1} + 10^{p-2} + \ldots + 10 + 1) = 6a_0$.

We must prove that $N(N+1) = 4(10^{2p-1}+10^{2p-2}+\ldots+10^p) + 2(10^{p-1} + \ldots + 10 + 1) = 4a_p + 2a_0$. We have $N(N+1) = 6a_0(6a_0+1) = 36a_0^2 + 6a_0$.

But $36 = 3 \cdot 10 + 6$ and a_0^2 may be written $(1+10+10^2+\ldots+10^{p-1})a_0$

$= a_0 + 10a_0 + \ldots + 10^{p-1}a_0 = a_0 + a_1 + a_2 + \ldots + a_{p-1}$.

Hence $N(N+1) = (3 \cdot 10+6)(a_0+a_1+\ldots+a_{p-1})+6a_0$

$= 3(a_1+a_2+\ldots+a_p)+6(a_0+a_1+\ldots+a_{p-1})+6a_0$

$= 9(a_1+a_2+a_3+\ldots+a_{p-1}) + 3a_p + 12a_0$

$= \left[10a_0+9(a_1+a_2+\ldots+a_{p-1})\right] + 3a_p + 2a_0$.

By definition of the a_0, a_1,..., we have $10a_0 = a_1$, $10a_1 = a_2$, etc. Therefore the expression in brackets reduces to just $10a_{p-1} = a_p$ and we have $N(N+1) = 4a_p + 2a_0$, which was to be proved.

* *

28. (1366, March 1935) Proposed by Norman Anning.

Show that the decimal fraction equivalent to the sum of the infinite series 1/10 + 2/100 + 3/1000 + ... does not contain the digit 8.

Solution:

$S = 1/10 + 2/100 + 3/1000 + \ldots$ and

$(1/10)S = 1/100 + 2/1000 + \ldots$. Subtracting,

$(9/10)S = 1/10 + 1/100 + 1/1000 + \ldots$

Therefore $(9/10)S = 1/9$, by use of the formula for the sum of an infinite geometric series. Thus $S = 1/9 \times 10/9 = 10/81 = 0.\overline{123456790}$,

by actual division. Hence, since S reduces to a repeating decimal with the period as given, no "8" appears.

* *

29. (1521, February 1938) Proposed by C. W. Trigg.

What is the smallest two-digit number which when expressed in another scale of notation merely has its digits reversed? What is the smallest three-digit number which behaves similarly?

Solution:

Let the numbers be converted from the systems of numeration with bases r and s into the decimal system of numeration. Then $ar + b = bs + a$, where if $r < s$ then $a > b$. Furthermore both a and b are less than r and greater than zero, so $r > 2$. From the equation $a(r-1) = b(s-1)$, if $r = 3$, then $a = 2$, $b = 1$, and $s = 5$. Hence the required number is 21 (base 3) = 12 (base 5). Similarly, $ar^2 + br + c = cs^2 + bs + a$, and if $r < s$, $a > c$. Since neither a nor c is zero, and no one of a,b,c is greater than or equal to r, then $r > 2$. Let $a = K+c$; then $K(r^2-1) = (s-r)\lfloor c(s+r)+b \rfloor$. When $r = 3$, and $s = 4$, $8K = 7c+b$, so $K = c = b = 1$. The desired number is accordingly,

$$211 \text{ (base 3)} = 112 \text{ (base 4)}.$$

Daniel Finkel notes that $13_{10} = 31_4$ and $371_{10} = 173_{16}$.

* *

30. (1616, December 1939) Proposed by C. W. Trigg.

Find the two permutations of the ten digits which are multiples of 1901 and are also perfect squares. Show that they are the only ones.

Solution:

Any number composed of the ten digits, 1,2,3,4,5,6,7,8,9,0 is a multiple of nine.

Let N equal the number whose square is the permutation. Then N is a multiple of 3 and also of 1901 or N is a multiple of 5703. Now N is not greater than the square root of 9876543210 or 99381 and N is not less than the square root of 1023456789 or 31991. Therefore N is a multiple of 5713 between the numbers 99381 and

31991. The first multiple of 5703 above 31991 is 34218. Hence, beginning with 34218 inspect the squares of the multiples of 5703 up to 95931.

There are only 11 such numbers. (57030 is omitted because of the final 0.)

$$45624^2 = 2081549376 \text{ and } 91248^2 = 8326197504$$

which are the two permutations sought.

* *

31. (1702, May 1941) Proposed by Paul C. Overstreet.

Show that for every perfect cube, n^3, there exists a series of n consecutive odd integers whose sum in n^3.

Solution:

Consider n^3 in the form $n \cdot n^2$.

If n is odd, say $n = 2k + 1$, form an arithmetic progression of n terms, with common difference 2, taking n^2 as the middle term. The progression would be $n^2 - 2k$, $n^2 - 2k + 2$, ..., $n^2 - 2$, n^2, $n^2 + 2$, ..., $n^2 + 2k - 2$, $n^2 + 2k$.

If n is even, say $n = 2j$, form a similar series except that $n^2 \pm 1$ are taken as the two middle terms. The progression would be n^2-2j+1, n^2-2j+3, ..., n^2-1, n^2+1, ..., n^2+2j-3, n^2+2j-1. In both cases the first term is $n^2 - n + 1$, and the last or nth term is $n^2 + n - 1$. The terms are n consecutive odd integers whose sum is n^3.

1. In School Science and Mathematics, May 1919, Oscar Schmiedel showed that any integral power of an integer is expressible as the sum of consecutive odd numbers, in an article entitled "Useful Benefits from the Study of Mathematics History."

2. It is clear from the above demonstration that the given property is not restricted to cubes or even to powers. Thus $m \cdot n$, with $m > n$, can be represented by the same method as a sum of n consecutive odd integers, if m and n are of the same parity.

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32. (1961, June 1946) Proposed by Brother Felix John.

Find a simple value for

$$\frac{(4+\sqrt{15})^{5/2}+(4-\sqrt{15})^{5/2}}{(6+\sqrt{35})^{5/2}-(6-\sqrt{35})^{5/2}} \qquad (1)$$

Solution:

$$4+\sqrt{15}=\left[\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right]^2 \qquad (2)$$

$$6+\sqrt{35}\quad\left[\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}}\right]^2 \qquad (3)$$

Inserting values of (2) and (3) in (1) it becomes

$$\frac{(\sqrt{5}+\sqrt{3})^5+(\sqrt{5}-\sqrt{3})^5}{(\sqrt{7}+\sqrt{5})^5-(\sqrt{7}-\sqrt{5})^5} \qquad (4)$$

Expanding, and simplifying expression (4) becomes:

$$\frac{11}{31}$$

* *

33. (1979, October 1946) Proposed by J. F. Arena.

Find the value of the infinite product: $3^{1/3} \cdot 9^{1/9} \cdot 27^{1/27} \ldots$

Solution:

The given expression may be written in the form

$$3^{1/3} \cdot 3^{2/9} \cdot 3^{3/27} \ldots$$

or

$$3^{(1/3+2/9+2/27+\ldots)}$$

Let

$$S = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \ldots + \frac{n}{3^n} + \ldots$$

Then

$$\frac{1}{3}S = \frac{1}{3^2} + \frac{2}{3^3} + \ldots + \frac{n-1}{3^n} + \ldots$$

$$(1-\frac{1}{3})S = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n} + \ldots$$

$$\frac{2}{3}S = \frac{1/3}{1-1/3} = \frac{1}{2}$$

$$S = \frac{3}{4}$$

Hence

$$3^{1/3} \cdot 9^{1/9} \cdot 27^{1/27} \cdot 81^{1/81} \ldots = 3^{3/4}$$

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34. (2040, October 1947) Proposed by Albert Hansen.

Prove that $x^2 - 3y = 17$ has no integral solution.

Solution:

Let

$$x = 3n \text{ (n an integer).}$$

Then

$$(3n)^2 - 3y^2 = 17$$

But this equation is impossible in integers since 3 is not a factor of 17. Thus

$$x \neq 3n \qquad (1)$$

Let

$$x = 3n \pm 1 \text{ (n an integer)}$$

Then

$$(3n \pm 1)^2 - 3y^2 = 17$$

and

$$9n^2 \pm 6n - 3y^2 = 16$$

But this equation is impossible in integers since 3 is not a factor of 16. Thus

$$x \neq 3n \pm 1 \qquad (2)$$

From (1) and (2), $x^2 - 3y^2 = 17$ has no integral solution.

Norman Anning suggests this approach:

Suppose integers x and y exist which satisfy the given equation.

Then $$x^2 - 2 = 3y^2 + 15 = 3(y^2 + 5)$$

This shows that the square of the integer x must leave the remainder 2 when it is divided by 3. Easy exploration shows that the squares of all integers leave only the remainders 0 or 1. No x can be found. The proof is complete.

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35. (2095, December 1948) Proposed by C. W. Trigg.

A nine-celled magic square is a 3 x 3 array in which the entries of each row, each column, and each diagonal have the same sum. Show that in a nine-celled magic square the central element must equal one-third of the constant. (The constant is the sum of the elements of any column, row, or diagonal.)

Solution:

The elements of the "general" nine-celled magic square are

a_1	b_1	c_1
a_2	b_2	c_2
a_3	b_3	c_3

its "constant" being S. Then the four sums involving the central element, b_2, are

$$a_1 + b_2 + c_3 = S$$

$$a_2 + b_2 + c_2 = S$$

$$a_3 + b_2 + c_1 = S$$

$$b_1 + b_2 + b_3 = S$$

Adding the above, grouping properly, we have $(a_1+a_2+a_3)+3b_2+(c_1+c_2+c_3)+(b_1+b_2+b_3) = 3S+3b_2 = 4S$, and $3b_2 = S$.

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36. (2105, January 1949) Proposed by Brother Felix John.

A man starts on a journey at 40 m.p.h. and reduces his speed by 2 miles every hour for an integral number of hours. If he had maintained the original rate, the journey would have taken 2 3/4 hours less. What was the distance traveled?

Solution:

Let d = distance traveled and x = number of hours required.

Then

$$d = 40 + 38 + 36 + 34 + \ldots + [40-2(x-1)] \qquad (1)$$

The right-hand member of (1), an arithmetic progression, may be summed as

$$d = \frac{x}{2}\{40 + [40-2(x-1)]\}$$

or reducing,

$$d = 41x - x^2 \qquad (2)$$

From the second sentence in the statement of the problem,

$$d = 40(x - \frac{11}{4}) \qquad (3)$$

Equating the values of d in (2) and (3) yields

$$41x - x^2 = 40x - 110$$

or

$$x^2 - x - 110 = 0$$

so

$$(x-11)(x+10) = 0$$

The only positive root of this equation is

$$x = 11 \text{ hours.}$$

Substituting in (3) gives

$$d = 330 \text{ miles.}$$

* *

37. (2194, October 1950) Proposed by Alan Wayne.

What are the two ways in which fifty coins are the correct change for a dollar?

Solution:

A half-dollar cannot be one of the coins, for then we would need 49 other coins to give a total value of fifty-cents obviously an impossibility. Hence, letting p be the number of pennies, n the number of nickels, d the number of dimes, and q the number of quarters, we have:

$$\begin{aligned} p + 5n + 10d + 25q &= 100 \\ p + n + d + q &= 50 \\ \hline 4n + 9d + 24q &= 50. \end{aligned}$$

Evidently the number of quarters cannot exceed one. If the number of quarters is one, we have $4n + 9d = 26$, which has the solution $n = 2$, $d = 2$. If there are no quarters, we have $4n + 9d = 50$, which has the solution $n = 8$, $d = 2$.

Hence we may have 45 pennies, 2 nickels, 2 dimes and 1 quarter; or 40 pennies, 8 nickels, and 2 dimes.

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38. (2701, April 1960) Proposed by C. W. Trigg.

The consecutive odd numbers are grouped as follows:

1, (3,5); (7,9,11); (13,15,17,19); ... Find the sum of the numbers in the nth group.

Solution:

It is clear that there are n odd numbers in the nth group. If T_n = the number of odd numbers which precede the nth group then

$$T_1 = 0,\ T_2 = 1,\ T_3 = 3, \text{ etc.}$$

$$\text{so } T_n = \frac{n(n-1)}{2}$$

Since the nth odd positive integer is 2n - 1 it follows that the first number in the nth group is

$$2\left[\frac{n(n-1)+2}{2}\right] - 1 = n(n-1) + 1$$

Using this result and the fact that the sum of the first n odd positive integers = n^2, the sum, S, of the nth group is readily obtained.

$$S = [n(n-1)+1]+[n(n-1)+3]+\ldots+[n(n-1)+(2n-1)]$$

$$S = n^2(n-1)+n^2$$

$$S = n^3$$

* *

39. (2728, November 1960) Proposed by Cecil B. Read.

Find a two-digit number equal to the square of the tens' digit plus the square of the sum of its digits.

Solution:

Let t be the tens' digit and u be the ones' digit. We have to solve $t^2 + (t+u)^2 = 10t + u$. Considering this as a quadratic in t, the roots are real if its discriminant $4[25-u(u+8)]$ is non-negative, which is true for u = 0,1,2; the roots are rational for u = 0 or 1 (when its discriminant is a perfect square), but irrational if u = 2. When u = 0, t = 5 (since $t \neq 0$) and when u = 1, t = 4. Thus the possible numbers are 50 and 41.

* *

40. (2945, November 1964) Proposed by Brother U. Alfred.

Determine the sum of all the positive proper fractions whose denominators are less than or equal to 100. (Note: Fractions need not be in their lowest terms. The series in question is 1/2 + 1/2 + 2/3 + 1/4 + 2/4 + 3/4 + 1/5 + ...).

Solution:

This series can be arranged as follows:

$1/2+(1/2+2/3)+(1/4+2/4+3/4)+(1/5)+\ldots)+\ldots = (1/2)(1+2+3+\ldots+99) =$

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since the $1+2+3+\ldots+n = n(n+1)/2$ where n is a positive integer.

* *

41. (3444, December 1972) Proposed by Cecil B. Read.

There are four whole numbers. Add 3 to the first, subtract 3 from the second, multiply the third by 3, divide the fourth by 3, and all are equal. If the difference between the two greater numbers is 36 more than the difference between the two smaller, what are the numbers?

Solution:

Let w, x, y, and z represent the four whole numbers. The problem conditions permit us to write

$$w + 3 = x - 3 = 3y = z/3 \qquad (1)$$

To establish an order relation among these four numbers we note from (1) that

$$w = x - 6 \text{ so } w < x \qquad z = 3w + 9 \text{ so } w < z$$
$$x = 3y + 3 \text{ so } y < x \qquad z = 9y \text{ so } y < z \qquad (2)$$

Therefore x and z are the two larger and w and y are the two smaller numbers.

Again from the problem conditions we have

$$|x - y| + 36 = |z - x| \qquad (3)$$

Substituting in (3) the following values obtainable from (1)

$$x = w + 6 \quad y = (w+3)/3 \quad z = 3w + 9 \qquad (4)$$

we have

$$|w - (w+3)/3| + 36 = |3w+9-w-6| \qquad (5)$$

Solving (5) we find $w = 24$ and from (4) the remainder of the solution set is $x = 30$, $y = 9$ and $z = 81$.

* *

42. (3559, April 1975) Proposed by C. W. Trigg.

The arithmetic mean of the twin primes 11 and 13 is the pentagonal number 12. Pentagonal numbers have the form $n(3n-1)/2$. Are there any other twin primes with a pentagonal mean?

Solution:

Twin primes > 3 have the forms 6k - 1 and 6k + 1, so their mean is 6k. If this is pentagonal, then $n(3n-1)/2 = 6k$, so that $6k - 1 = (3n^2-n-2)/2 = (3n+2)(n-1)/2 = f(n)$.

Clearly, if n is odd, f(n) is composite except when n = 3, whereupon 6k - 1 = 11.

If n = 2m, then f(n) = (3m+1)(2m-1). This obviously is composite for m > 1, and if m = 1, f(n) = 4. Clearly 4 is not a prime.

Therefore, 12 is the unique pentagonal mean of twin primes.

* *

43. (3681, February 1978) Proposed by Joe Dan Austin.

Find numbers n(2), n(3), ..., n(11) so that for i = 2, 3, ..., 11 each n(i) (a) uses each of the digits 0, 1, 2, ..., 9 exactly once; (b) is divisible by i; and (c) is the largest number satisfying (a) and (b).

Solution:

The largest number formed from the base ten digits is 9,876,543,210. Using standard divisibility tests, the number is found to be divisible by 2, 3, 9, 5, and 10. Since the number is divisible by 2 and 3, it is divisible by 6.

By looking at the second largest number, 9,876,543,201, to be formed from the ten digits, we find 7 divides it. Neither of the preceding two numbers is divisible by 4 or 8. But the third largest number, 9,876,543,120, is divisible by both 4 and 8. Using the alternating sign test for divisibility by 11, the alternating sum of the largest number is -5. By switching pairs of digits, the sum becomes divisible by 11. The largest number which contains each of the decimal digits exactly once with a sum divisible by 11 is 9,876,524,130. The results are summarized in the table below.

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i	n(i)
2	9,876,543,210
3	9,876,543,210
4	9,876,543,120
5	9,876,543,210
6	9,876,543,210
7	9,876,543,201
8	9,876,543,120
9	9,876,543,210
10	9,876,543,210
11	9,876,524,130

* *

44. (3699, October 1978) Proposed by Herta T. Freitag.

Let n be a given positive integer. If S_1 denotes the sum of the odd positive integers smaller than n and S_2 represents the analogous sum comprised of even positive integers, determine $|S_1 - S_2|$.

Solution:

When n = 2k,

$$S_1 = 1 + 3 + \ldots + (2k-3) + (2k-1)$$

and

$$S_2 = 2 + 4 + \ldots + (2k-2).$$

Thus

$$|S_1 - S_2| = \underbrace{|(-1)+(-1)+\ldots+(-1)}_{(k-1)\text{ addends}}+(2k-1)| =$$

$$|(-1)(k-1)+(2k-1)| = |k| = k.$$

When n = 2k - 1,

$$S_1 = 1 + 3 + \ldots + (2k-3)$$

and

$$S_2 = 2 + 4 + \ldots + (2k-2).$$

Hence

$$|S_1 - S_2| = \underbrace{|(-1)+(-1)+\ldots+(-1)|}_{(k-1)\text{ addends}} =$$

$$|(-1)(k-1)| = |k-1| = k - 1.$$

Putting k in terms of n,

$$|S_1 - S_2| = \begin{cases} n/2 \text{ if } n \text{ is even} \\ (n-1)/2 \text{ if } n \text{ is odd} \end{cases} = \lfloor n/2 \rfloor, \text{ where } \lfloor m \rfloor \text{ denotes}$$

the greatest integer which does not exceed m.

* *

45. (3705, November 1978) Proposed by Alan Wayne.

Show that for each positive integer n there is an n-digit positive integer N such that N^2 starts with precisely n ones.

Solution:

We note first that

$$4^2 = 16,$$
$$34^2 = 1156,$$
$$334^2 = 111556,$$

and so on.

Thus we conjecture that

$$N = 33333...4$$

where there are n-1 threes preceding the final 4.

To prove this conjecture, we see that N can be represented as $\frac{10^n + 2}{3}$, and so $N^2 = \frac{10^{2n+4} \cdot 10^{n+4}}{9} = \frac{10^{2n}-1}{9} = \frac{4(10^n-1)}{9} + \frac{9}{9}$.

Now clearly $\frac{10^{2n}-1}{9}$ will consist of 2n ones, and $\frac{4(10^n-1)}{9}$ of n fours. Thus

$$N^2 = 11...1111...11$$
$$\quad\quad\quad\quad\quad 44...44$$
$$+ \quad\quad\quad\quad\quad\quad\quad 1$$
$$11...1155...56,$$

where each of the first n digits is a 1, each of the next n-1 digits is a 5, and the last digit is a 6.

* *

NOTES